ELSEVIER Journal of Computational and Applied Mathematics 57 (1995) 45–56

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Central limit theorem by polynomial dependence coefficients

René Blacher

LMC-IMAG, Equipe SMS, B.P. 53x, 38041 Grenoble Cedex, France

Received 23 October 1992; revised 20 February 1993

Abstract

We study the asymptotic behaviour of normalized sums of n random variables $\sum_n = n^{-1/2}(X_1 + X_2 + \cdots + X_n)$. We obtain a necessary and sufficient condition in order that the moments of $\sum_n$ converge to the moments of a normal distribution. This condition is expressed by means of polynomial dependence coefficients $\rho_{j_1, j_2, \ldots, j_n}$. Then, one can obtain central limit theorems with minimal assumptions.

Keywords: Orthogonal polynomials; Central limit theorem; Fourier transform

1. Introduction

Let $\{X_n\}$ be a sequence of real random variables defined on a probability space $(\Omega, \mathscr{A}, \boldsymbol{P})$. Probabilists have wanted to know the distributions of $X_1 + X_2 + \cdots + X_n$ and $(X_1)^2 + (X_2)^2 + \cdots + (X_n)^2$. At first, this problem has been studied with the X_j's independent. Then, one has tried to weaken this assumption. In this paper, we shall notice that orthogonal polynomials can give a strong answer to this problem.

For sake of simplicity, we suppose that the X_j's have the same law m with $\mathbb{E}\{X_1\} = 0$ (where $\mathbb{E}\{\cdot\}$ is expectation). Then, we denote by $\{P_j\}_{j \in \mathbb{N}}$ the family of orthonormal polynomials associated to m.

When the X_j's are independent, usual proofs use the Fourier transform [11, 12].

Notations 1.1. Let M be a probability on $\mathbb{R}^h$, $h \in \mathbb{N}^*$. Let $g \in L^2(\mathbb{R}^h, M)$. We define $\phi_{g,M}$ by

$$\phi_{g,M}(t) = \int e^{it(x_1 + x_2 + \cdots + x_h)} g(x_1, x_2, \ldots, x_h) M(dx_1, dx_2, \ldots, dx_h).$$

Then, we set $\phi_n = \phi_{\mu_n}$, where μ_n is the law of $(X_1, X_2, \ldots, X_n)$.

In this paper, we represent by ε any real sequence which converges to 0 as $n \to \infty$. Then, by derivation of $\phi_{P_j \cdot m}$, one proves easily that the Fourier transforms of the P_j's have the following property.

0377-0427/95/\$09.50
SSDI 0377-0427(93)E0233-C

Theorem 1.2. *For all $j \in \mathbb{N}$, $\phi_{P_j \cdot m}(t) = (\sigma_j/j!)(\mathrm{i}t)^j + \mathrm{o}(|t|^j)$ where $\sigma_j = \mathbb{E}\{X_1^j P_j(X_1)\}$. Moreover, let $I \in \mathbb{N}$. Then, for all $j \in \mathbb{N}$,*

$$\phi_{P_j \cdot m}\left(\frac{t}{\sqrt{n}}\right) = \frac{\sigma_j}{j!}\left(\frac{\mathrm{i}t}{\sqrt{n}}\right)^j \left(1 + \sum_{k=1}^{I-j} \varepsilon_{j,k}^n (\mathrm{i}t)^k + \mathrm{o}\left(\left|\frac{t}{\sqrt{n}}\right|^{I-j}\right)\right).$$

So, orthogonal polynomials fit well to this study. As a matter of fact, the above property is more convenient for the study of limit distributions.

For exact distributions, we can use the Hermite or Laguerre polynomials. Indeed, let $\{H_j\}$ and $\{L_j^a\}$ be the families of orthonormal polynomials associated to $N(0,1)$ and $\gamma(a,2)$, respectively, where $N(m,\sigma^2)$ is the normal distribution on $\mathbb{R}$ with mean m and variance σ^2, and $\gamma(a,\lambda)$ is the gamma distribution with parameters a and λ, $a > 0$. Then, we have the following equalities [15, p. 376, Eq. (19)], [9, p. 312], [8].

Theorem 1.3. *For all $j \in \mathbb{N}$,*

$$\phi_{H_j N(0,1)}(t) = \frac{(\mathrm{i}t)^j \mathrm{e}^{-t^2/2}}{\sqrt{j!}} \quad \textit{and} \quad \phi_{L_j^a \cdot \gamma(a,2)}(t) = \sqrt{\frac{\Gamma(a+j)}{j!\,\Gamma(a)}}\,\frac{(2\mathrm{i}t)^j}{(1-2\mathrm{i}t)^{a+j}}.$$

Then, by using this result, we obtain the exact distribution of a sum of n random variables. Indeed, we suppose that $(X_1, X_2, \dots, X_n)$ has a density belonging to L^2 with respect to $N(0,1)^{\otimes} = N(0,1) \otimes N(0,1) \otimes \cdots \otimes N(0,1)$, the product normal distribution on $\mathbb{R}^n$. Then, we can obtain f and g, the density with respect to $N(0,n)$ and $\gamma(n/2,2)$, of $X_1 + X_2 + \cdots + X_n$ and $(X_1)^2 + (X_2)^2 + \cdots + (X_n)^2$, respectively.

In this purpose, we define the complex random vector $(Z_1, Z_2, \dots, Z_n)$ by $Z_s = \mathrm{i}U_s + X_s$, where $(U_1, U_2, \dots, U_n)$ is independent of $(X_1, X_2, \dots, X_n)$ and has the distribution $N(0,1)^{\otimes}$. Then, it is shown that the density of $X_1 + X_2 + \cdots + X_n$ is equal to

$$f(y) = \sum_{j=0}^{\infty} \mathbb{E}\left\{\frac{[Z_1 + Z_2 + \cdots + Z_n]^j}{\sqrt{n^j j!}}\right\} H_j\left(\frac{y}{\sqrt{n}}\right),$$

and that the density of $X_1^2 + X_2^2 + \cdots + X_n^2$ is equal to

$$g(y) = \sum_{j=0}^{\infty} \mathbb{E}\left\{\frac{[Z_1^2 + Z_2^2 + \cdots + Z_n^2]^j}{\sqrt{2j!}}\right\} \mathbb{E}\{H_{2j}\{\sqrt{By}\}\},$$

where B is a random variable which has the Beta distribution with parameters $\frac{1}{2}$ and $\frac{1}{2}(n-1)$ (i.e. $\mathbb{E}\{H_{2j}((By)^{1/2})\}$ is a Laguerre polynomial).

As a matter of fact, these results are still valid in the vector case (i.e. for the random vectors $\{X_{s,1} + X_{s,2} + \cdots + X_{s,n}\}$ and $\{(X_{s,1})^2 + (X_{s,2})^2 + \cdots + (X_{s,n})^2\}$). These results are proved in [3].

Therefore, these results generalize some key theorems of the probability theory. If we compare them with the above results, we notice efficiency of the Fourier transform of orthogonal polynomials in this study [12, pp. 133, 158–162, 171–173], [6, 8, 14].

In this paper we use the same method for limit distributions. We set $\sum_n = n^{-1/2}(X_1 + X_2 + \cdots + X_n)$. By the famous central limit theorem, we know that $\sum_n$ has asymptotically the distribution $N(0, \sigma^2)$ if the X_j's are independent and $\sigma^2 = \mathbb{E}\{(X_n)^2\} < \infty$. The generalization of this result for the no-independent case has been treated in many papers. Generally, these papers use the Ibragimov's theorems on the strong mixing sequences [10].

At first, we need the following notations.

Notations 1.4. For all $(j_1, j_2, \ldots, j_n) \in \mathbb{N}^n$, we set $\rho_{j_1, j_2, \ldots, j_n} = \mathbb{E}\{P_{j_1}(X_1) \cdot P_{j_2}(X_2) \cdots P_{j_n}(X_n)\}$ and $\alpha_{j_1, j_2, \ldots, j_n} = \mathbb{E}\{\tilde{P}_{j_1}(X_1) \cdot \tilde{P}_{j_2}(X_2) \cdots \tilde{P}_{j_n}(X_n)\}$, where $\tilde{P}_j = \{\sigma_j/(j!)\} P_j$.

Then, the convergence of moments is equivalent to a condition on $\alpha_{j_1, j_2, \ldots, j_n}$'s.

Theorem 1.5. *All the moments $M_q^n = \mathbb{E}\{(X_1 + X_2 + \cdots + X_n)^q / n^{q/2}\}$ converge to $M_q \in \mathbb{R}$ if and only if, for all $q \in \mathbb{N}$, there exists $S_q \in \mathbb{R}$ such that*

$$(q! n^{-q/2}) \sum_{\substack{j_1 + j_2 + \cdots + j_n = q \\ j_s \leqslant 2}} \alpha_{j_1, j_2, \ldots, j_n}$$

converges to S_q.

For the convergence to a normal distribution, we can specify this theorem.

Theorem 1.6. *For all $q \in \mathbb{N}$, M_q is the moment of order q of a distribution $N(0, M_2)$, if and only if for all $q \in \mathbb{N}$, S_q is the moment of order q of a distribution $N(0, S_2)$. In this case, $M_2 = S_2 + (\sigma_1)^2$.*

The interest of these theorems is that the $\rho_{j_1, j_2, \ldots, j_n}$'s are indeed dependence coefficients. For example, $\rho_{j_1, j_2, \ldots, j_n} = 0$ if one of the X_j's is independent of the other ones.

As a matter of fact, the $\rho_{j_1, j_2, \ldots, j_n}$'s measure polynomial dependences and each one measures a particular type of dependence. In particular, when $n = 2$, ρ_{j_1, j_2} is the polynomial correlation coefficient of order (j_1, j_2) between X_1 and X_2. For example, $\rho_{1,1}$ is the classical correlation coefficient: $\rho_{1,1}$ measures the linear dependence.

As a matter of fact, by using the $\rho_{j_1, j_2, \ldots, j_n}$'s, we can have a complete study of dependence. The most interesting property of these coefficients is that they can detect the most of the functional dependences [1, 3, 13].

On the other hand, Theorems 1.5 and 1.6 give only an equivalence to the convergence of moments. In other words, we only turn this convergence of moments into a condition on the dependence coefficients $\rho_{j_1, j_2, \ldots, j_n}$. Therefore, in these theorems, there is no asymptotical independence assumption. Besides, we can easily build up some sequences $\{X_n\}$ whose moments converge without the fact that the X_j's are independent. For example, let us take $X_n = e_n Y$ where Y has a normal distribution and $e_n = \pm 1$ is correctly chosen.

As a matter of fact, in order to have asymptotical independence conditions, it is enough to choose assumptions a little stronger on the $\rho_{j_1, j_2, \ldots, j_n}$'s. Clearly, by this method, we shall obtain minimal assumptions for the central limit theorem. For example, we have the following theorem.

Theorem 1.7. *We suppose that*

$$n^{-2} \sum_{\substack{j_1+j_2+\cdots+j_n=4 \\ j_s=2 \text{ or } 0}} \rho_{j_1,j_2,\ldots,j_n}$$

converges to 0. *We suppose also that for all* $q \in \mathbb{N}^*$,

$$n^{-q/2}(q!)\left(\sum_{t_1=1}^{n} \sum_{t_2=t_1+1}^{n} \cdots \sum_{t_q=t_{q-1}+1}^{n} \mathbb{E}\{X_{t_1} X_{t_2} \cdots X_{t_q}\}\right)$$

converges to the moment of order q *of* $N(0, S_2)$, *and that*

$$n^{-q/2} \sum_{\substack{j_1+j_2+\cdots+j_n=q \\ j_s \leqslant 2,\ \text{only one } j_s=2}} \rho_{j_1,j_2,\ldots,j_n}$$

is bounded.

Then, $\sum_n$ *has asymptotically the normal distribution* $N(0, M_2)$ *with* $M_2 = S_2 + (\sigma_1)^2$.

In order to better appreciate the interest of these results, we are going to compare them with the classical theorems on the strong mixing processes. For this purpose, we need the following notations.

Notations 1.8. We set $\hat{\sigma}_n^2 = \mathbb{E}\{(X_1 + X_2 + \cdots + X_n)^2\}$. Moreover, in the Theorems 1.9 and 1.11, we define exceptionally $\sum_n$ by $\sum_n = (X_1 + X_2 + \cdots + X_n)/\hat{\sigma}_n$.

Let $r(n)$ be an increasing sequence such that $r(n) \in \mathbb{N}$, $r(1) = 0$, $r(n) \leqslant n$, and $[r(n)]/n$ converges to 0 as $n \to \infty$. We define the sequences $u(n)$ and $t(n)$ by: $u(1) = 1$, $t(1) = 0$, $u(n) = \text{Max}\{m \in \mathbb{N}^*: 2m + r(m) \leqslant n\}$ and $t(n) = n - 2u(n)$ when $n \geqslant 2$.

Then, we set $\sum_u = (X_1 + X_2 + \cdots + X_u)/\hat{\sigma}_u$, $\sum_u' = (X_{u+t+1} + X_{u+t+2} + \cdots + X_{2u+t})/\hat{\sigma}_u$ and $\xi_u = (X_{u+1} + X_{u+2} + \cdots + X_{u+t})/\hat{\sigma}_u$.

We suppose that $\{X_n\}$ is strictly stationary and that $\mathbb{E}\{\xi_u^2\}$ converges to 0 as $n \to \infty$.

As a matter of fact, we have just decomposed $\sum_n$ in two terms $\sum_u$ and $\sum_u'$, and ξ_u which separates them is negligible. Then, we can choose asymptotical independence conditions between $\sum_u$ and $\sum_u'$. In the following theorem, these conditions are written using moments [4].

Theorem 1.9. *We suppose that, for all* p *and* $q \in \mathbb{N}^*$, $\mathbb{E}\{(\sum_u)^p(\sum_u')^q\} - \mathbb{E}\{(\sum_u)^p]\mathbb{E}[(\sum_u')^q\}$ *converges to* 0 *as* $n \to \infty$.

Then, all the moments of $\sum_n$ *converge to the moments of* $N(0, 1)$.

Clearly this theorem has stronger assumptions than Theorem 1.7.

Remark 1.10. In Theorem 1.5, we have chosen a denominator equal to $n^{1/2}$ in order not to have complicated statements. However, we can generalize this result without big difficulties if we replace $n^{1/2}$ by another function. Anyway, it does not change anything to the main result of this paper: we can still obtain a *logical equivalence* with the convergence of moments.

Now, in usual central limit theorems, the asymptotical independence conditions are generally written by using sets. Moreover, results on moments are theorems which assert that moments converge under some assumptions [16].

So, it is difficult to compare well classical results with above theorems. For example, on the one hand, we suppose that moments exist. On the other hand, the assumptions on dependence are much stronger than a necessary and sufficient condition.

However, the following theorem allows us to have a better understanding of respective interest of these results. Indeed, Theorem 1.11 corresponds to Theorem 1.9, but with asymptotical independence conditions which are written by using sets [5].

Theorem 1.11. *We suppose that there exists a sequence* $\pi_k \to 0$ *such that* $\mathbb{E}_{|\sum_n| \geq k}\{(\sum_n)^2\} \leq \pi_k$.

We set $\mathscr{T}_{k,p,q} = [p + 4^{-k}q, p + 4^{-k}(q+1)[$, $p \in \mathbb{Z}$, $k, q \in \mathbb{N}$, $0 \leq q \leq 4^k$. *We suppose that, for all* k, p, p', q, q', $(p, q) \neq (p', q')$, $\boldsymbol{P}\{\{\sum_u \in \mathscr{T}_{k,p,q}\} \cap \{\sum'_u \in \mathscr{T}_{k,p',q'}\}\} - \boldsymbol{P}\{\sum_u \in \mathscr{T}_{k,p,q}\} - \boldsymbol{P}\{\sum'_u \in \mathscr{T}_{k,p',q'}\}$ *converges to* 0 *when* $n \to \infty$.

Then, $\sum_n$ *is asymptotically distributed as the normal distribution* $N(0, 1)$.

As a matter of fact, Theorems 1.5–1.7 have been proved in 1989 [2]. Then, these results have suggested a weaker theorem: Theorem 1.9. As the form of this one was more usual, it has been published earlier in order to appreciate better the results of [2]. At the same time, the theorems for exact distributions have been written in [3]. Then, we have proved Theorem 1.11 in order to have assumptions using sets.

Now, this last theorem has also independence hypotheses much weaker than those known so far. For example, for the strong mixing sequences, for *all* Borelian set $\mathscr{B}$ and $\mathscr{B}'$, $|\boldsymbol{P}\{\{\sum_u \in \mathscr{B}\} \cap \{\sum'_u \in \mathscr{B}'\}\} - \boldsymbol{P}\{\sum_u \in \mathscr{B}\}\boldsymbol{P}\{\sum'_u \in \mathscr{B}'\}| \leq \alpha(t(n))$.

Now, if we suppose also that $(\hat{\sigma}_n)^2 = \hat{\sigma}n(1 + o(1))$, then $\sum_n$ converges if and only if [11, p. 339]

$$\lim_{k \to \infty} \left\{ \limsup_{n \to \infty} \{\mathbb{E}_{|\Sigma_n| \geq k}\{\Sigma_n^2\}\} \right\} = 0.$$

Finally, in Theorem 1.7, orthogonal polynomials have therefore allowed to obtain very interesting forms of central limit theorems.

We have to emphasize that it is the results of this paper which have suggested Theorems 1.9 and 1.11 (the proof of Theorem 1.11 is the simple adaptation of the Ibragimov's one). Indeed orthogonal polynomials give a more simple presentation of some probabilist's problems: it is not their least interest.

Remark 1.12. (i) We can obtain Theorem 1.5 without using orthogonal polynomials. But it is not the same when denominators $\hat{\sigma}_n$ are different from $n^{1/2}$. As a matter of fact, the purpose of this paper is to let people know the method used herein before we finalize some new results about other stable distributions.

(ii) All these results are detailed in [2].

2. Notations of Fourier transforms

At first, we recall that one can write characteristic functions by using moments.

Theorem 2.1. *For all* $I \in \mathbb{N}^*$,

$$\phi_n\left(\frac{t}{\sqrt{n}}\right) = \sum_{k=0}^{I} M_k^n \frac{(\mathrm{i}t)^k}{k!} + \mathrm{o}(|t|^I).$$

So, we need series expansion limited at order I in order to prove the convergence of the moments.

Notations 2.2. We set

$$f_n^q(x_1, x_2, \ldots, x_n) = \sum_{j_1 + j_2 + \cdots + j_n = q} \rho_{j_1, j_2, \ldots, j_n} P_{j_1}(x_1) P_{j_2}(x_2) \cdots P_{j_n}(x_n),$$

$$f_n^{[I]} = \sum_{q=0}^{I} f_n^q, \qquad \phi_n^q = \phi_{f_n^q . m^{\otimes}} \quad \text{and} \quad \phi_n^{[I]} = \phi_{f_n^{[I]} m^{\otimes}},$$

where $m^{\otimes}$ is the product measure $m \otimes m \otimes \cdots \otimes m$ on $\mathbb{R}^n$.

Let $M_n^{[I]}$ be the measure which has the density function $f_n^{[I]}$ with respect to $m^{\otimes n}$. Then, $M_n^{[I]}$ has the same moments of order $(j_1, j_2, \ldots, j_n)$ as $(X_1, X_2, \ldots, X_n)$ if $j_1 + j_2 + \cdots + j_n \leqslant I$:

$$\phi_n^{[I]}(t) - \phi_n(t) = \mathrm{o}(|t^I|). \tag{1}$$

Therefore, in order to study the convergence of the moments, it is enough to study each ϕ_n^q.
For example, $\rho_{0,0,\ldots,0} = 1$. Therefore,

$$\phi_n^{[0]}\left(\frac{t}{\sqrt{n}}\right) = \left(\phi_m\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

which is the characteristic function of $n^{-1/2}(\tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_n)$ where the $\tilde{X}_j$'s are independent and have the law m. Therefore, by the central limit theorem

$$\left(\phi_m\left(\frac{t}{\sqrt{\sigma_1^2 n}}\right)\right)^n$$

converges to $\mathrm{e}^{-t^2/2}$. Moreover, we know that the moments of $n^{-1/2}(\tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_n)$ converge to the moments of $N(0,1)$ (cf, for example, [16]). Then, we have the following theorem.

Theorem 2.3. *In this paper, we denote by* μ_k *the moments of the normal distribution* $N(0, \sigma_1^2)$. *Moreover, we denote by* μ *indexed by* n *and* k *the sequences which converge to* μ_k *as* $n \to \infty$.
Then, with these notations,

$$\left[\phi_m\left(\frac{t}{\sqrt{n}}\right)\right]^{n-q} = 1 + \sum_{k=2}^{I} \mu_{q,k}^n \frac{(\mathrm{i}t)^k}{k!} + \mathrm{o}(|t|^I).$$

Now, we study ϕ_n^1. Clearly, $\rho_{j_1, j_2, \ldots, j_n} = 0$ if a single $j_s \neq 0$. Therefore, $\phi_n^1 = 0$, $\phi_n^{[I]}(t/\sqrt{n}) = \phi_n^{[0]}(t/\sqrt{n})$, and $M_1 = 0$.

3. Lemmas

At first, we introduce the following notations.

Notation 3.1. Let $\mathscr{S}_q^*$ the substitution of $\{\{j_1, j_2, \ldots, j_n\} \in \mathbb{N}^n | j_1 + j_2 + \cdots + j_n = q\}$ defined by $\mathscr{S}_q^*(j_1, j_2, \ldots, j_n) = \{u_1, u_2, \ldots, u_n\}$ where $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_n$.

We suppose $n > q$ because $n \to \infty$. Then, $u_{q+1} = u_{q+2} = \cdots = u_n = 0$ and we define $\mathscr{S}_q$ by $\mathscr{S}_q(j_1, j_2, \ldots, j_n) = \{u_1, u_2, \ldots, u_q\}$ $(q > 0)$.

Let $\mathscr{P}_q = \{\mathscr{S}_q(j_1, j_2, \ldots, j_n) | j_1 + j_2 + \cdots + j_n = q\}$. Let $\mathscr{O}_q \in \mathscr{P}_q$. By misuse of our notations, we set $u_s \in \mathscr{O}_q = \{u_s \in \mathbb{N} | s = 1, 2, \ldots, q, \{u_1, u_2, \ldots, u_q\} = \mathscr{O}_q\}$.

We set

$$S_{\mathscr{O}_q}^n = (q!)\left[\frac{\sum_{\mathscr{S}_q(j_1, j_2, \ldots, j_n) = \mathscr{O}_q} \alpha_{j_1, j_2, \ldots, j_n}}{\sqrt{n^q}}\right] \quad \text{and} \quad S_q^n = (q!)\left[\frac{\sum_{j_1 + j_2 + \cdots + j_n = q} \alpha_{j_1, j_2, \ldots, j_n}}{\sqrt{n^q}}\right].$$

Then, we have the following property.

Lemma 3.2. *For all* $q \in \mathbb{N}^*$,

$$\phi_n^q\left(\frac{t}{\sqrt{n}}\right) = S_q^n \frac{(\mathrm{i}t)^q}{q!} + \mathrm{o}(|t|^q).$$

Moreover, if S_q^n *converges to* S_q, *then,*

$$\phi_n^q\left(\frac{t}{\sqrt{n}}\right) = S_q \frac{(\mathrm{i}t)^q}{q!} + \mathrm{o}(|t|^q).$$

Proof. We know

$$\phi_n^q\left(\frac{t}{\sqrt{n}}\right) = \sum_{\mathscr{O}_q \in \mathscr{P}_q}\left\{\sum_{\mathscr{S}_q(j_1, \ldots, j_n) = \mathscr{O}_q} \rho_{j_1, j_2, \ldots, j_n}\left(\prod_{u_s \in \mathscr{O}_q}\left[\phi_{P_{u_s}, m}\left(\frac{t}{\sqrt{n}}\right)\right]\right) \cdot \left[\phi_m\left(\frac{t}{\sqrt{n}}\right)\right]^{n-q}\right\}.$$

By Theorems 1.2 and 2.3,

$$\left[\prod_{u_s \in \mathscr{O}_q} \phi_{P_{u_s}, m}\left(\frac{t}{\sqrt{n}}\right)\right]\left[\phi_m\left(\frac{t}{\sqrt{n}}\right)\right]^{n-q} = \left(\prod_{u_s \in \mathscr{O}_q} \frac{\sigma_{u_s}}{u_s!}\right)\left(\frac{\mathrm{i}t}{\sqrt{n}}\right)^q + \mathrm{o}(|t|^q).$$

Therefore,

$$\phi_n^q\left(\frac{t}{\sqrt{n}}\right) = S_q^n \frac{(\mathrm{i}t)^q}{q!} + \mathrm{o}(|t|^q). \qquad \square$$

Now, we have also the following lemma.

Lemma 3.3. *Let $q \in \mathbb{N}$ such that $S^n_{\mathscr{C}_q}$ is bounded for all $\mathscr{C}_q \in \mathscr{P}_q$. Then, for every $I \in \mathbb{N}^*$,*

$$\phi^q_n\left(\frac{t}{\sqrt{n}}\right) = S^n_q \frac{(\mathrm{i}t)^q}{q!}\left[1 + \sum_{k=1}^{I} \mu_k \frac{(\mathrm{i}t)^k}{k!}\right] + (\mathrm{i}t)^q \left[\sum_{k=1}^{I} \varepsilon^{*n}_{q,k} (\mathrm{i}t)^k\right] + \mathrm{o}(|t|^{q+I})$$

Moreover,

$$\phi^q_n\left(\frac{t}{\sqrt{n}}\right) = S_q \frac{(\mathrm{i}t)^q}{q!}\left[1 + \sum_{k=1}^{I} \mu_k \frac{(\mathrm{i}t)^k}{k!}\right] + (\mathrm{i}t)^q \left[\sum_{k=0}^{I} \varepsilon^{n}_{q,k} (\mathrm{i}t)^k\right] + \mathrm{o}(|t|^{q+I})$$

if S^n_q converges to S_q.

Proof. By Theorems 1.2 and 2.3, we can also write

$$\left[\prod_{u_s \in \mathscr{C}_q} \phi_{P_{u_s},m}\left(\frac{t}{\sqrt{n}}\right)\right]\left[\phi_m\left(\frac{t}{\sqrt{n}}\right)\right]^{n-q} = \left(\prod_{u_s \in \mathscr{C}_q} \frac{\sigma_{u_s}}{u_s!}\right)\left(\frac{\mathrm{i}t}{\sqrt{n}}\right)^q \left[1 + \sum_{k=1}^{I} [\mu^n_{\mathscr{C}_q}]_k \frac{(\mathrm{i}t)^k}{k!} + \mathrm{o}(|t|^I)\right].$$

Therefore,

$$\phi^q_n\left(\frac{t}{\sqrt{n}}\right) = \sum_{\mathscr{C}_q \in \mathscr{P}_q} S^n_{\mathscr{C}_q} \frac{(\mathrm{i}t)^q}{q!}\left[1 + \sum_{k=1}^{I} [\mu^n_{\mathscr{C}_q}]_k \frac{(\mathrm{i}t)^k}{k!} + \mathrm{o}(|t|^I)\right]. \quad \square$$

Now, we need the following notations.

Notations 3.4. Let h, k and r be three integers such that $1 \leqslant r \leqslant k \leqslant h$. We set

$$t_1 \neq t_2 \neq \cdots \neq t_r, t_{r+1}, \ldots, t_k = \{(t_1, t_2, \ldots, t_r, t_{r+1}, \ldots, t_k) \in \{1, 2, \ldots, n\}^k \mid t_s \neq t_{s'} \text{ if } s < s' \leqslant r\},$$

$$t_1 \neq t_2 \neq \cdots \neq t_{r-1}, t_{r+1}, \ldots, t_k = \{(t_1, t_2, \ldots, t_{r-1}, t_{r+1}, \ldots, t_k) \in \{1, 2, \ldots, n\}^{k-1} \mid t_s \neq t_{s'} \text{ if } s < s' \leqslant r-1\}.$$

In particular, $(t_1, t_2, \ldots, t_k) = \{1, 2, \ldots, n\}^k$.

Then, we can prove the following property.

Lemma 3.5. *For all $s \in \{1, 2, \ldots, k\}$, we denote by R_s a polynomial of degree j_s such that $R_s(x) = P_1(x) = x/\sigma_1$ if $j_s = 1$. We set $\sum_{s=1}^{k} j_s = h$. We set also $k_1 = \operatorname{card}\{j_s = 1\}$, $k_2 = \operatorname{card}\{j_s = 2\}$ and $k_3 = \operatorname{card}\{j_s > 2\}$. We define H^* by $H^* = h - 1$ if $k_3 = 0$ and $k_2 \leqslant 1$, and $H^* = h - 2$ if not. We define H by $H = H^*$ if H^* is even and $H = H^* + 1$ if not. We set $B_h = \max\{1, |M^n_h|\}$.*

Then

$$\left|\mathbb{E}\left[\frac{\sum_{t_1 \neq \cdots \neq t_r, t_{r+1}, \ldots, t_k} \left(\prod_{s=1}^{k} R_s(X_{t_s})\right)}{\sqrt{n^h}}\right]\right| \leqslant c(n) \cdot B_H + K_3 |M^n_h|,$$

with $c(n) = K_1$ when $k_3 = 0$, and $c(n) = n^{-1/2} K_2$ when $k_3 > 0$, where K_1, K_2, K_3 are three constants with $K_3 = 0$ if there exists s such that $j_s \geqslant 2$.

Proof. We prove this property by recurrence on r. At first, we prove it for $r = 1$ when there exists s such that $j_s > 1$ (the result is obvious if not).

By Holder's inequality,

$$\left| \mathbb{E}\left\{ \frac{\sum_{t_1, t_2, \dots, t_k} \left(\prod_{s=1}^{k} R_s(X_{t_s}) \right)}{\sqrt{n^h}} \right\} \right| = \left| \mathbb{E}\left\{ \prod_{s=1}^{k} \left[\frac{\sum_{t=1}^{n} R_s(X_t)}{\sqrt{n^{j_s}}} \right] \right\} \right|$$

$$\leqslant \left[\mathbb{E}\left\{ \left| \frac{\sum_{t=1}^{n} X_t}{\sigma_1 \sqrt{n}} \right|^H \right\} \right]^{k_1/H} \left[\prod_{j_s = 2} \mathbb{E}\left\{ \left| \frac{\sum_{t=1}^{n} R_s(X_t)}{n} \right|^H \right\} \right]^{1/H}$$

$$\times \left[\prod_{j_s > 2} \mathbb{E}\left\{ \left| \frac{\sum_{t=1}^{n} R_s(X_t)}{\sqrt{n^3}} \right|^H \right\} \right]^{1/H} [\mathbb{E}(1)]^{1-(k_1+k_2+k_3)/H}.$$

Moreover,

$$\mathbb{E}\left\{ \left(\frac{\sum_{t=1}^{n} R_s(X_t)}{n} \right)^H \right\} \leqslant \frac{\sum_{t_1, t_2, \dots, t_H} \mathbb{E}\{[R_s(X_1)]^H\}}{n^H} = \mathbb{E}\{[R_s(X_1)]^H\} < +\infty.$$

Now we suppose that Lemma 3.5 holds for all $k \leqslant h$ and all $r' \leqslant r - 1$. Then,

$$\mathbb{E}\left[\frac{\sum_{t_1 \neq t_2 \neq \cdots \neq t_{r-1}, t_r, \dots, t_k} R_1(X_{t_1}) R_2(X_{t_2}) \cdots R_k(X_{t_k})}{\sqrt{n^h}} \right]$$

$$= \mathbb{E}\left[\frac{\sum_{t_1 \neq t_2 \neq \cdots \neq t_r, t_{r+1}, \dots, t_k} R_1(X_{t_1}) R_2(X_{t_2}) \cdots R_k(X_{t_k})}{\sqrt{n^h}} \right]$$

$$+\ \mathbb{E}\left[\frac{\sum_{t_1 \neq t_2 \neq \cdots \neq t_{r-1}, t_{r+1}, \ldots, t_k} [R_1(X_{t_1}) R_r(X_{t_1})] \cdot R_2(X_{t_2}) \cdots R_{r-1}(X_{t_{r-1}}) R_{r+1}(X_{t_{r+1}}) \cdots R_k(X_{t_k})}{\sqrt{n^h}}\right]$$

$$+\ \mathbb{E}\left[\frac{\sum_{t_1 \neq t_2 \neq \cdots \neq t_{r-1}, t_{r+1}, \ldots, t_k} R_1(X_{t_1}) \cdot [R_2(X_{t_2}) R_r(X_{t_2})] \cdots R_{r-1}(X_{t_{r-1}}) R_{r+1}(X_{t_{r+1}}) \cdots R_k(X_{t_k})}{\sqrt{n^h}}\right]$$

$$+ \cdots .$$

Then, it is enough to apply the recurrence assumption. □

4. Proofs of the theorems

4.1. Proof of the sufficiency condition of Theorem 1.5

We prove by recurrence on h that, for all $q \leqslant h$, M_q^n and S_q^n converge, and that $S_{\mathcal{O}_q}^n$ is bounded for all $\mathcal{O}_q \in \mathcal{P}_q$.

We have already studied ϕ_n^0 and ϕ_n^1. Clearly $S_0^n = 1$ and $S_1^n = 0$, and the property holds for $q \leqslant 1$. *So, we suppose that it holds for all* $q \leqslant h - 1$.

Let $\mathcal{O}_h = (u_1, u_2, \ldots, u_h)$ with $u_1 \geqslant 3$. By Lemma 3.5,

$$\mathbb{E}\left[\frac{\sum_{t_1 \neq t_2 \neq \cdots \neq t_k} \tilde{P}_{u_1}(X_{t_1}) \tilde{P}_{u_2}(X_{t_2}) \cdots \tilde{P}_{u_k}(X_{t_k})}{\sqrt{n^h}}\right]$$

converges to 0. Therefore, $S_{\mathcal{O}_h}^n$ converges to 0.

Due to our assumption, we know that the sum of the $S_{\mathcal{O}_h}^n$'s such that $u_1 \leqslant 2$ converges. Therefore S_h^n converges also. Then, by Theorem 2.1, Eq. (1), and Lemmas 3.2 and 3.3, we deduce that M_h^n converges.

Then, we deduce from Lemma 3.5 that all the $S_{\mathcal{O}_h}^n$'s are bounded.

4.2. Proof of the necessity condition of Theorem 1.5

Now, we suppose that all the moments converge.

At first, we prove that the S_h^n's converge. For $h = 0, 1$, this result is obvious.

So, we suppose that, for all $q \leqslant h - 1$, S_q^n converges to S_q. By Lemma 3.5, for all $p \in \mathbb{N}^*$, all the $S_{\mathcal{O}_p}^n$'s are bounded. Therefore, we can apply Lemma 3.3. Then, from Theorem 2.1, Eq. (1) and Lemma 3.3, we deduce that S_h^n converges.

It is enough to use Lemma 3.5 again in order to complete the proof.

4.3. Proof of Theorem 1.6

We suppose that the M_h^n's converge (or, by Theorem 1.5, the S_h^n's). By Lemma 3.5, we deduce that the $S_{\mathscr{C}_q}$'s are bounded. Therefore, we can apply Lemma 3.3.

Then, the expansion limited at order h of

$$\phi_n^{[h]}\left(\frac{t}{\sqrt{n}}\right)$$

converges to

$$\sum_{q=0}^{h}\sum_{k=0}^{h-q} S_q \mu_k \frac{(\mathrm{i}t)^{q+k}}{(q!)(k!)}.$$

Therefore,

$$M_h = \sum_{t=0}^{h} \frac{(h!)}{[(h-t)!](t!)} S_{h-t}\,\mu_t.$$

Therefore, if we know the S_q's, we know also the M_h's, and conversely. Then, we can easily verify that, if S_q is the moment of order q of $N(0, S_2)$, M_q is the moment of order q of $N(0, \mu_2 + S_2)$.

We remark that is possible that $S_2 < 0$. In this case, the S_q's are the moments of $\mathrm{i}Y$, where Y has the distribution $N(0, -S_2)$.

4.4. Proof of Theorem 1.7

At first, we prove by recurrence on h that all the M_h^n's are bounded. We suppose that this property holds for all $q \leqslant h-1$. By Lemma 3.5, $S_{\mathscr{C}_q}^n$ and S_q^n are bounded when $q < h$.

By Lemma 3.5 again, $S_{\mathscr{C}_h}^n$ is bounded if $\mathscr{C}_h = (u_1, u_2, \dots, u_h)$ with $u_1 \geqslant 3$ or $u_2 \geqslant 2$. Therefore, by the hypothesis, all the $S_{\mathscr{C}_h}^n$'s are bounded. Therefore S_h^n is bounded.

Therefore, all the S_q^n and $S_{\mathscr{C}_q}^n$ are bounded for $q \leqslant h$. We deduce from Theorem 2.1 and Lemmas 3.2 and 3.3 that M_h^n is bounded. *Therefore all the moments are bounded.*

Now we need the following lemma.

Lemma 4.1. *We suppose $R_1 = P_2$. Then, for all r, $1 \leqslant r \leqslant k$,*

$$n^{-h/2}\,\mathbb{E}\left\{\sum_{t_1 \neq t_2 \neq \cdots \neq t_r, t_{r-1}\dots, t_k}\left[\prod_{s=1}^{k} R_s(X_{t_s})\right]\right\}$$

converges to 0.

Proof. By the Schwartz inequality,

$$n^{-h}\left(\mathbb{E}\left\{\sum_{t_1, t_2, \dots, t_k}\left[\prod_{s=1}^{k} R_s(X_{t_s})\right]\right\}\right)^2$$
$$\leqslant \mathbb{E}\left\{n^{-2}\left(\sum_{t=1}^{n} R_1(X_t)\right)^2\right\}\mathbb{E}\left\{n^{2-h}\left[\prod_{s=2}^{k}\left(\sum_{t_s=1}^{n} R_s(X_{t_s})\right)\right]^2\right\}.$$

By Lemma 3.5, the term on the right-hand side is bounded.

Moreover,

$$n^{-2}\mathbb{E}\left\{\left(\sum_{t=1}^{n}\mathbb{R}_1(X_t)\right)^2\right\}-n^{-2}\sum_{t=1}^{n}\mathbb{E}\{P_2(X_t)^2\}=n^{-2}\sum_{t_1\neq t_2}\mathbb{E}\{P_2(X_{t_1})P_2(X_{t_2})\}$$

which converges to 0 by assumption. We deduce the lemma if $r = 1$.

When $r > 1$, we prove the result by recurrence by using the same equality as in Lemma 3.5. □

Therefore, $S^n_{\mathcal{O}_h}$ converges to 0 for $\mathcal{O}_h = (u_1, u_2, \dots, u_h)$ with $u_1 = 2$. By Lemma 3.5, it is the same when $u_1 \geqslant 3$. Then, all the S^n_h's converge to the S_h's. Therefore, it is enough to apply Theorem 1.6 and the moments theorem [7, p. 108, Section 2.1].

References

[1] R. Blacher, Coefficients de corrélation d'ordre supérieur, *Statist. Anal. Données* **9** (2) (1984) 48.

[2] R. Blacher, Loi de la somme de n variables aléatoires et théorème de la limite centrale par les coefficients de dépendance polynômiale et les moments, LMC-IMAG Research Report 768-M, 1989; submitted for publication.

[3] R. Blacher, Quelques applications des fonctions orthogonales en probabilité et statistiques, Thèse de l'Université Joseph Fourier, 1990.

[4] R. Blacher, Théorème de la limite centrale par les moments, *C. R. Acad. Sci Paris Sér. I Math.* **311** (1990) 465–468.

[5] R. Blacher, Une nouvelle forme pour le théorème de la limite centrale, LMC-IMAG Research Report 852-M, 1991; *Ann. Inst. H. Poincaré*, submitted.

[6] D.W. Boyd, N.L. Johnson and S. Kotz, Series representations of distributions of quadratic forms in normal variables II, non central case, *Ann. Math. Statist.* **38** (1967) 838–848.

[7] D.A.S. Fraser, *Nonparametric Methods in Statistics* (Wiley, New York, 1957).

[8] R.A. Gideon and J. Gurland, Series expansions for quadratic forms in normal variables, *J. Amer. Statist. Assoc.* **71** (353) (1976) 227–232.

[9] T. Hida, *Brownian Motion* (Springer, Berlin, 1980).

[10] I.A Ibragimov, Some limit theorems for stationary processes, *Theory Probab. Appl.* **4** (1962) 349–382.

[11] I.A. Ibragimov and Yu.V. Linnik, *Independent and Stationary Sequences of Random Variables* (Wolters-Noordhoff, Groningen, 1971).

[12] N.L. Johnson and S. Kotz, *Distribution in Statistics. Continuous Univariate Distributions, Vol. 2* (Wiley, New York, 1970).

[13] H.O. Lancaster, Orthogonal models for contingency tables, in: P.R. Krishnaiah, Ed., *Developments in Statistics* (Academic Press, New York, 1980).

[14] H.E. Robbins, The distribution of a definite quadratic form, *Ann. Math. Statist.* **19** (1948) 266–270.

[15] G. Szegő, *Orthogonal Polynomials* (Amer. Mathematical Soc., Providence, RI, 1959).

[16] R. Yokoyama, The convergence of moments in the central limit theorem for stationary φ-mixing processes, *Anal. Math.* **9** (1983) 79–84.